MODERN ALGEBRA

Short Questions on Modern Algebra

Determine whether each of the following statements is true or false Justify your answer. (Each question will carry two marks.)

1. If $f: A \to B$ and $g: B \to C$ are two mappings and if $g \circ f: A \to C$ is bijective, then g is injective.
2. Let (G, o) be a group and $a, b \in G$, $b \neq e$ (e is the identity element of (G, o)) If $o(a) = 3$ and $aba^{-1} = b^2$ then $o(b) = 7$.
3. Klein's 4-groups K_4 is a cyclic group.
4. The set $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R) \mid a+c = b+d \right\}$ is a subring of the ring $M_2(R)$ of all 2×2 real matrices.
5. If U has an inverse in a ring R with unity, then $-U$ has also an inverse where $U \in R$.
6. In a field, the equation $ax = b$ has unique solution, where $a, b \in F$ and $a \neq 0$.
7. The group $(Q, +)$ of rational numbers under addition is a cyclic group.
8. If a ring contains a left devisor of zero, it contains a both-sided divisor of zero.
9. The field $(Z_3, +, *)$ is of characteristic 3
10. For non-empty set S, $S \times S$ is an equivalence relation on S.
11. The mapping $f: R \to (-1, 1)$ defined by $f(x) = \frac{x}{1+x^2}$, $x \in R$ is a bijective mapping.
12. In a group (G, o), if $\forall a \in G$, $a^{-1} = a$, then (G, o) is abelian.

13. Every group of order less than 4 is commutative.

14. In a cyclic group $(G, \circ)$ if g_1, g_2 are two distinct generators then $g_1 \circ g_2$ is also a generator of $(G, \circ)$

15. The ring $(Z_6, +, \cdot)$ of all integers modulo 6 is an integral domain.

16. In a ring $(R, +, \cdot)$ if $a^2 = a \ \forall a \in R$, then $a + a = 0$

17. In a field F, $(-a)^{-1} = -(a^{-1}) \ \forall a \in F \ (a \neq 0)$.

18. If $Q[\sqrt{2}] = \{a + b + \sqrt{2} \quad a, b \in Q\}$, then the ring $(Q[\sqrt{2}], +, \cdot)$ is a field.

19. If $S = \{1, 2, 3, \ldots, 10, 11, 12\}$, then the sets ϕ, $\{1, 3, 5, 8\}$, $\{2, 4, 6, 9\}$, $\{5, 9, 11, 12\}$ is a partition of S.

20. On the set of inters, a relation ρ is defined by $a \rho b$ iff $ab \geq 0$; the relation ρ is reflexive and transitive.

21. S is the set of 2×2 real matrices and binary operation $\circ$ is defined on S by $A \circ B = (AB + BA)$ for $A, B \in S$, then $(S, \circ)$ is a semigroup.

22. If Z^* is the set of non-negative integers and $f : Z \to Z^*$ is defined by $f(x) = \frac{1}{2}\{x + |x|\}$, then $f(x)$ is bijective mapping

23. Any two left cosets of a subgroup of a group are either disjoint or identical.

24. If $R\sqrt{3} = \{a + b\sqrt{3} \quad a, b \in R\}$, then the ring $(R\sqrt{3}, +, \cdot)$ is a field

25. Union of two proper subrings of a ring is also a subring.

26. If $G = \{x \quad x^{15} = 1\}$, then all the proper subgroups of $(G, \cdot)$ are cyclic

27. A, B, C are three nonempty sets and if $A \subset B \subset C$, then $A' \subset A' \subset C'$

28. $f.\ Z \to Z$; $g : Z \to Z$ given by $f(x) = x + 1$, $g(x) = 2x + 3$; $g \circ f$ is a surjective mapping.

29. Let $(G,\circ)$ be a group. $\forall\, x, y \in G$, if $xy = yz \Rightarrow x = z$, then $(G,\circ)$ is abelian.

30. $M_2 = \left\{ \begin{pmatrix} 2^m & 0 \\ 0 & 3^n \end{pmatrix} : m, n \text{ are nonzero integers} \right\}$. $(M_2, \cdot)$ forms a group with respect to matrix multiplication $(\cdot)$.

31. The ring Z_6 of all integers modulo 6 is an integral domain

32. Let $(F, +, \cdot)$ be a field If $a, b \in F$ such that $b \neq 0$ and $(ab)^2 = ab^2 + bab - b^2$, then $a = 1$

33. Any finite ring with unity is a field.

34. A group $(G, *)$ is abelian if $b^{-1} * a^{-1} * b * a = e$ where $a, b \in G$ and e is the identity of $(G, *)$.

35. The mapping $f : Z \to Z$ (Z is the set of integers), defined by $f(x) = x + 2$ is not surjective.

36. The identity element is the only idempotent element in a group

37. There exist a subgroup with 5 elements in a group of 8 elements

38. If R be a ring and a be a fixed element of R, then the set $S = \{ x \; x \in R,\ x \cdot a = 0 \}$ is a subring of R.

39. If x be an idempotent element of a ring R with identity 1, then $1 - x$ is also idempotent in R.

40. In a field, the equation $a \cdot x = b$ has unique solution, where $a, b \in F$ and $a \neq 0$.

41. The symmetric group S_3 is a commutative group.

42. In a group $(G,\circ)$, if $a = a^{-1}$ $\forall a \in G$, then G is a commutative group

43. The additive group of all rational numbers is a cyclic group

44. Every field is an integral domain.

45. There exist a field with 7 elements.

46. The set $T = \{a + b\sqrt{2} \mid a, b \text{ are rational numbers}\}$ is a subfield of the field of rational numbers.

47. Let R^* denote the set of all non-zero real numbers. Define a binary operation '$\circ$' on R^* by $a \circ b = |a| b \ \forall a, b \in R^*$. Then $(R^*, \circ)$ is a group

48. The matrix $M_2(R)$ has zero divisors.

49. In a group $(G, \cdot)$, if $a \cdot b = a \cdot c$ then $b = c$

50 In a group $(G, \cdot)$, if $(a \cdot b)^2 = a^2 \cdot b^2 \ \forall a, b \in G$ then $(G, \cdot)$ is abelian.

51. If H, K be two subgroups of a group $(G, \cdot)$, then $H \cup K$ is also a subgroup of $(G, \cdot)$

52. In a ring R, $a^2 - b^2 = (a+b) \cdot (a-b)$ for all $a, b \in R$

53. In a ring R, $(a-b) - c = a - (b+c) \ \forall a, b, c \in R$.

54. In a field $(-a)^{-1} = -(a^{-1}) \ \forall a \in F$ and $a \neq 0$

55. A subring with identity of a field is an integral domain

56. If $f: A \to B$ is invertible, then $f^{-1}: B \to A$ is also invertible

57. The intersection of two subgroups of a group is a subgroup

58. Every group of order 6 is a commutative group.

59. In a ring R, $a \cdot (-b) = (-a) \cdot b \ \forall a, b \in R$.

60. In a ring R, if $(a+b)^2 = a^2 + 2ab + b^2 \ \forall a, b \in R$, then R is commutative.

61. If ρ is a symmetric and transitive relation on a set S, then ρ is a reflexive relation.

62. If R is a ring and $e \in R$ such that $e^2 = e$, then $(x \cdot e - exe)^2 = (ex - exe)^2 \ \forall x \in R$.

63. Let $(H_1, \circ)$ and $(H_2, \circ)$ be two subgroups of a group $(G, \circ)$. If $H_1 \cup H_2$ is a subgroup of G and $H_1 \not\subseteq H_2$, then $H_2 \subseteq H_1$

64. Any finite commutative group is a cyclic group

Sets, Mappings & Relations

1. If A, B, C are subsets of a universal set S, prove that

(i) $[A \cap (B \cup C)] \cap [A' \cup (B' \cap C')] = \varphi$

(ii) $(A - B) \times C = (A \times C) - (B \times C)$

(iii) If $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then $B = C$

(iv) $[A \cup (B' \cap C')]' = \varphi$

(v) $A \times (B - C) = (A \times B) \cap (A \times C')$

(vi) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(vii) if $A \cap C = B \cap C$ and $A \cap C' = B \cap C'$ then $A = B$

(viii) $(A \cap B^{\circ})' = (A' \cup C')$

(ix) $(A \cap C) \cup (B \cap C') = \varphi \Rightarrow A \cap B = \varphi$

(x) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(xi) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(xii) $S - (A \cup B) = (S - A) \cap (S - B)$

(xiii) $(A \cap B) - C = (A - C) \cap (B - C)$

(xiv) $A - (B \cap C) = (A - B) \cup (A - C)$

(xv) $(A \cup B)' = A' \cap B'$

2. (a) Define partition of a non-empty set.

(b) Define equivalence relation on a non-empty set

(c) Define injective, surjective and bijective mappings

(d) Define equivalence Class

3 Give in each case an example of a mapping which is

(i) Reflexive but neither symmetric nor transitive.

(ii) Symmetric but neither reflexive nor transitive.

(iii) Transitive but neither reflexive nor symmetric

(iv) Reflexive but neither symmetric nor transitive.
(v) Symmetric but neither reflexive nor transitive.
(vi) Transitive but neither reflexive nor symmetric.

4. Prove the following:

(a) A partition of a set induces an equivalence relation on that set.

(b) If ρ ~~be~~ is an equivalence relation on a nonempty set S, then ρ will determine a partition on S

(c) Let R be an equivalence relation on a non-empty set S and $a, b \in S$. ~~Prove That~~ $[a] = [b]$ iff aR_b

(d) Let ρ be an equivalence relation on a set A and $a, b \in A$ such that $(a, b) \notin \rho$ Then $cl(a) \cap cl(b) = \varphi$.

(e) If $f: A \to B$ be a mapping and P, Q be two non-empty subsets of A Then $f(P \cup Q) = f(P) \cup f(Q)$ Give an example where $f(P \cap Q) \neq f(P) \cap f(Q)$.

(f) Let $f: A \to B$ and g $B \to C$ be two bijective mappings. If $g \circ f$ $A \to C$ bijective, prove that f is injective and g is bijective.

(g) Let $f: N^+ \to N^+$ be the mapping given by $f(n) = 2x$, when N^+ is the set of all positive integers. Then f is injective but not surjective.

(h) Let $f: A \to B$ be an injective mapping from a set A into the set B. If C and D are subsets of A, then $f(C \cap D) = f(C)$

(i) Let $f: A \to B$, $g: B \to C$, $h: B \to C$ be three mappings such that f is surjective and $g \circ f = h \circ f$. Then $g = h$.

I. Solve the following problems:

(a) Let $A = R - \{-\frac{1}{2}\}$, $B = R - \{\frac{1}{2}\}$ where R denotes the set of all real numbers. Let $f : A \to B$ be defined by $f(x) = \frac{x-3}{2x+1}$, $\forall x \in A$. Does f^{-1} exist. Justify.

(b) Let Z^* be the set of all non-zero integers and let $S = Z \times Z^*$ (Z being the set of integers). Let $R = \{x \mid x = ((r,s),(t,u)) \in S \times S$ with $ru = st\}$. Prove that R is an equivalence relation on S.

(c) A relation ρ on the set of integers Z is defined by $\rho = \{(a,b) \in Z \times Z \; ; \; |a-b| \le 5\}$. Is the relation reflexive, symmetric and transitive?

(d) A function $f : Z^* \to Z$ is defined by

$f(n) = \frac{n}{2}$ where n is even integer.

$= \frac{-n+1}{2}$ where n is odd integer

Is f injective and surjective? (Z^* is the set of negative integers and Z is the set of integers.)

(e) Let $f : N^+ \to N^+$ be the mapping given by $f(n) = 2n$, where N^+ is the set of all positive integers. Show that f is injective but not surjective.

(f) Which of the following relations is an equivalence relation on the set of integers?

(i) $P = \{(x,y) \in Z \times Z \mid xy \geq x\}$

(ii) $Q = \{(x,y) \in Z \times Z \mid x - y$ is a multiple of $5\}$

Give reasons in support of your answer.

(g) Find all the bijective mappings from the set A onto itself where $A = \{1, 2, 3\}$.

(h) Let S be the set of all 2×2 real matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad-bc \neq 0$ and R^* denote the set of all non-zero real numbers. Show that the mapping $f: S \to R^*$ defined as follows: $f\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc$, is surjective but not injective.

(i) In each case examine whether the relation ρ is an equivalence relation on the set S of all integers.

(i) $\rho = \{(a, b) \in S\times S \quad |a-b| \leq 3\}$

(ii) $\rho = \{(a, b) \in S\times S \quad a-b$ is a multiple of $6\}$

(j) Show that the following relation $\rho = \{(a, b) \in Z\times Z :$ $a-b$ divisible by $5\}$ is an equivalence relation on the set Z of all integers. Find all distinct equivalence classes of

(k) Find the equivalence relation corresponding to the partition $Z_1 \cup Z_2$ of Z where Z is the set of all integers Z_1 is the set of all odd integers and Z_2 is the set of all even integers.

(l) Let Q be the set of all rational numbers. Prove that the mapping $f: Q \to Q$ defined by $f(x) = 5x+2$ $(x \in Q)$ is a bijective mapping. Find f^{-1}

(m) Let H be a subgroup of a group G. Show that the relati $\rho = \{(a, b) \in G\times G \quad a^{-1}b \in H\}$ is an equivalence relation on the set G

Group

1. Prove the following:

(a) Any finite sem-group in which both the cancellation laws hold is a group. Is it true in case of infinite semigroup? Justify.

(b) Each element of a finite group is of finite order.

(c) Every group of even order contains an element of order 2.

(d) Every subgroup of a cyclic group is cyclic.

e) Let $(G,\circ)$ be a ~~group~~ group and $(H,\circ)$ be a subgroup of $(G,\circ)$. Let $a, b \in G$ Then prove that $aH = bH$ iff $a^{-1}b \in H$.

(f) Let $(G,*)$ be a group. A nonempty subset S of G forms a subgroup of $(G,*)$ iff $a \in S$, $b \in S \Rightarrow a*b^{-1} \in S$.

(g) Let $(G,*)$ be a group and $(S_1,*)$, $(S_2,*)$ are two subgroups of $(G,*)$. Then $S_1 \cup S_2$ form a subgroup of $(G,*)$ iff either $S_1 \subset S_2$ or $S_2 \subset S_1$

(h) Order of a subgroup of a finite group divides the order of the group

(i) Every group of prime order is cyclic

(j) $(S,\cdot)$ is a semigroup. Show that $(S,\cdot)$ will form a subgroup if $a\cdot x = b$ and $y\cdot a = b$ have unique solution in S for any two $a, b \in S$

(k) If g is a generator of a cyclic group, show that g^{-1} is also as generator of it.

(l) State and prove Lagrange's theorem for a finite group

(m) Any finite semigroup in which both cancellation laws hold is a group.

(n) Every cyclic group is abelian

(o) ~~[illegible]~~ A group (G, o) of order n is cyclic iff there exists an element of order n in (G, o).

(p) A non-empty subset H of a group (G,o) is a sub-group of (G, o) iff $\forall\, h, k \in H$, $h^{-1}k \in H$.

2 Solve the following:

(a) Every group of even order contains an element of order

(b) In a symmetric group S_4 of degree 4, solve the equat $x \circ (1, 2, 3) = (2, 4\ 3)$; ['o' represents the composition of permutation

(c) Let (G, o) be a group and $a, b \in G$ If $(a \circ b)^3 = a^3 \circ b^3$ $\forall a, b \in G$, show that $H = \{x^3 \quad x \in G\}$ is a subgroup of $(G,$ w.r.t. 'o'

(d) Prove that the set S is a group under matrix multiplic where S is the set of all 2×2 real matrices $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with ac: Is it commutative?

(e) $(S, \circ)$ is a semi-group. If for all $x, y \in S$, $x^2 \circ y = y = y \circ x^2$, prove that $(S, \circ)$ is an abelian group.

(f) Show that the set of 2×2 real matrices forms a group with respect to matrix multiplication. Show that this group is not abelian.

(g) In a finite group $(G, \cdot)$, show that (i) a and xax^{-1} are of the same order for $a, x \in G$ (ii) if for $a \in G$, $a^3 = e$ (identity in G) and $aba^{-1} = b^2$, then order of a is 7.

(h) Let G be a group. If $a^{-1}b^2(bab)^{-1}ba^2 = b$, $\forall a, b \in G$, then show that G is a commutative group.

(i) Let $S = \{(a, b) \mid a \neq 0, a, b \in R\}$ and the operation $\circ$ be defined on S by $(a, b) \circ (c, d) = (ac, bc + d)$. Show that $(S, \circ)$ forms a group. Is it commutative?

(j) Let $G = \{2^n \mid n \in Z, Z = \text{set of integers.}\}$ Prove that G forms a cyclic group w.r.t. multiplication. Find the generators of this group.

(k) Let $G = \{z \mid z \in C, |z| = 1\}$ where C is the set of all complex numbers. Show that G is a commutative group w.r.t. usual multiplication in C.

(l) Show that the group of rational numbers under usual addition is not a cyclic group.

(m) Let $G = \left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \text{ is an integer}\right\}$ Show that G is a group w.r.t. usual matrix multiplication. Is this group commutative?

(n) Let a, b, c be three elements of a group G. Find an element x of G such that $(axb)^{-1} = c^{-1}b^{-1}$ Is such an element x unique? Justify.

(o) Show that the set G of all complex numbers $a+ib$ such that $a^2+b^2=1$, is a group w.r.t. usual complex multiplication.

(p) Show that the set G of all ordered pairs (a, b) with $a \neq 0$ of real numbers a, b is a group with operation $*$ defined by $(a, b) * (c, d) = (ac, bc+d)$.

(q) If x, y are elements of a group G, prove that
$$y^{-1}x^{-2}y = (y^{-1}xy)^{-2}$$

(r) Let $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R} \text{ and } ad-bc \neq 0 \right\}$.
Prove that M becomes a non-commutative group under usual matrix multiplication.

(s) Let G be the set of all 2×2 matrices $\begin{bmatrix} a & b \\ -b & d \end{bmatrix}$ where a, b are real numbers and are not zero simultane Is G a group w.r.t. matrix multiplication? Justify.

(t) If a, b be two elements of a group G such that
$$b^2ab = a^{-1}, \text{ then show that } (ba)^3 = a.$$

(u) If $a(ab)^2b = a^3b^3$ for all $a, b \in G$, then G is a commutative group

(v) Show that a group G is commutative iff $\forall a, b \in G$, $(ab)^2 = a^2b^2$

Ring and Fields

1. Prove the following:

(a) The field Q of rational numbers has no proper sub-fields.

(b) Let $S = \{ a + b\sqrt{5} \mid a, b$ are real numbers$\}$. Prove that S is commutative ring with 1 under usual addition and multiplication. Is this a field? Justify.

(c) The characteristic of an integral domain is either zero or a prime number

(d) Prove that a finite integral domain is a field. Is the converse true? Justify.

(e) Let R be a ring. Prove that

(f) $a \cdot 0 = 0 \cdot a = a$ and $(-a) \cdot (-b) = ab \ \forall a, b \in R$.

(g) Prove that a commutative ring R satisfies cancellation property for multiplication iff R has no zero divisors.

(h) Let R be a ring with identity element 1 and containing at least two distinct elements. Prove that $1 \neq 0$

(i) Prove that a field does not contain any divisors of zero.

(j) ~~In~~ Let $(R, +, \cdot)$ be a field and a, b, c be three elements of R such that $a \neq 0$. Then there exists one and only one value of x in R such that $ax + b = c$

(K) A commutative ring $(R, +, \cdot)$ with unity is an integral domain iff the cancellation law holds for multiplication in R.

(l) Cancellation law fails to hold in the Z_6 (the ring of integers modulo 6).

(m) In a ring $(R,+,\cdot)$ if (i) $a^2=a$ $\forall a\in R$ then $b=-b$ $\forall b\in$ and (ii) $(a+b)^2 = a^2+2ab+b^2$ for any $a,b\in R$ then R is commutative ring.

(n) Intersection of two subrings of a ring $(R,+,\cdot)$ is a subring of $(R,+,\cdot)$.

(o) A finite commutative ring (containing more than one element) without zero divisors is a field.

2. (a) Let T be a ring with unity I, having no divisors of zero and if for $a,b\in T$, $ab=1$, then show that $ba=$

(b) Show that the field of rational numbers has no proper subfield.

(c) Let $S=\{a+b\sqrt{5} \mid a,b$ are real numbers$\}$. Prove that S is a commutative ring with unity under usual addition and multiplication. Is this a field? Justify.

(d) In a ring $(R,+,\cdot)$, show that $(a-b)-c = a-(b+c)$ $\forall a,b,c\in R$.

(e) In a field $(F,+,\cdot)$, prove that $a^2=b^2 \Rightarrow$ either $a=b$ or $a=-b$; $a,b\in R$.

(f) Prove that the ring $(Z_p,+,\cdot)$, where p is a prime, is an integral domain. Is it a field? Justify.

(g) In a ring $(R,+,\cdot)$, show that (i) if $a,b\in R$ and $m,n\in Z^+$ (set of positive integers), then $(ma)(nb)=mn(a$

(h) $(R,+,\cdot)$ is a ring with unity, if a is a unit in R, then show that a is not a divisor of zero.

(i) If C denotes the set of complex numbers, show that the additive abelian group $(C,+)$ forms a ring $(C,+,\cdot)$ with unity under usual addition and multiplication. Does it form a field? Justify.

(j) Prove that the ring of matrices $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \middle| a, b \in Q\right\}$ is a field. Is this result true if $a, b \in R$? Justify.

(K) $(R,+,\cdot)$ is a commutative ring with unity 1 having no divisors of zero and if for $a, b \in R$, $ab = 1$, then show that $ba = 1$

(l) Show that the set $T = \{(a,0) \mid a \in Z\}$ is a subring of the ring $Z \times Z$ where Z is the ring of integers and addition & multiplication in $Z \times Z$ are defined by $(a,b) + (c,d) = (a+c, b+d)$, $(a,b)\cdot(c,d) = (ac, bd)$. Is the identity element of T equal to the identity element of $Z \times Z$? Give reasons.

(m) Let R be a commutative ring with unity. Prove that R is an integral domain iff for a non-zero element $x \in R$, $x\cdot m = x\cdot n \Rightarrow m = n$ where $m, n \in R$.

(n) Let S denote the set of all functions $f: R \to R$ where R denotes the field of real numbers. Define $+$ and $*$ on S by $(f+g)(a) = f(a) + g(a)$ and $(f*g)(a) = f(a)\,g(a)$ for all $a \in R$ and for all $f, g \in S$. Assuming S to be an additive abelian group, show that $(S,+,*)$ is a commutative ring with ~~iden~~ unity. Is this ring a field

(o) Prove that characteristic of an integral domain is either 0 or a prime number

(p) Prove that the ring of matrices $\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in R \right\}$ is a field

(q) Prove that the set F of all matrices of the form $\begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ forms a field w.r.t the usual matrix multiplication and addition where a, b are any two rational numbers.

(r) If n is a ~~positive~~ prime number, then prove that the ring Z_n of all integers modulo n is a field. Is the converse true? Justify.

(s) In a ring $(R, +, \cdot)$ prove that $a \cdot 0 = 0 \cdot a = 0$ and $(-a)\cdot(-b) = ab \ \forall a, b \in R$

(t) In the field R of all real numbers, Show that the set $A = \{ a + b\sqrt{3} \in R \mid a, b \in Q \}$ is a subfield, but the set $B = \{ b\sqrt{3} \in R \mid b \in Q \}$ is not a subfield

(u) Prove that the set of all 2x2 matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b \in R$ is not a field w.r.t. matrix addition and multiplication

(v) In the field $(Z_7, +, \cdot)$ of all integers modulo 7 find $([5] + [6])^{-1}$ and $(-[4])^{-1}$ where $[4], [5], [6] \in Z_7$

(w) Let $\Gamma = Q \times Q = \{ (a, b) \mid a, b \in Q$, the set of rational numbers$\}$. In Γ define addition '+' and multiplication '$\cdot$' by, $(a, b) + (c, d) = (a+c, b+d)$ and $(a, b)\cdot(c, d) = (ac, bd)$.
Show that Γ is a commutative ring but not a field