Question Bank (For Mathematics-~~Honours~~)

Department of Mathematics

Classical Algebra

A. **Integers** :

1. State first and second principle of induction 1+1
2. Using first principle of induction prove that
(i) $2+7+12+ \cdots + (5n-3) = \frac{n(5n-1)}{2}$ 3
(ii) $2 \cdot 3^n + 5^{2n-1}$ is divisible by 11 ~~for~~ for all $n \in N$.
(iii) $10^n + 10^{n-1} + 1$ is divisible by 3, for all $n \in N$
3. Using second principle of induction prove that $(5+\sqrt{7})^n + (5-\sqrt{7})^n$ is an even integer for all $n \in N$. 3
4 Prove that $3^n - 5^n - 8^n + 10^n$ is divisible by 10 for all $n \in N$ 3
5 If p be an odd prime number then prove that $8 \mid p^2 - 1$ 4
6. If $2^n - 1$ be a prime number prove that n is prime 3
7 If p and p^2+8 are both prime numbers, prove that $p = 3$ 3
8. Use Euclidean algorithm to find integers x and y such that $\gcd(36, 69) = 36x + 69y$. 2
9 Find the general solution in integers and the least positive integral solutions of the equations 4
(i) $39x - 23y = 25$ (ii) $31x - 13y = 10$
10. The sum of two positive integers is 100. If one is divided by 4, the other is divided by 5 and the remainder is 4 in each case. Find the numbers 4
11. Find the number of positive divisors $\tau(n)$ of a positive integer n where 2
(i) $n = 472$ (ii) $n = 1842$ (iii) $n = 2921$
12 If $n = 2^{k-1}(2^k - 3)$ where $k > 1$ and $2^k - 3$ is prime then show that sum of all positive divisors of n, $\sigma(n) = 2n + 2$

13. If $a \equiv b \pmod{m}$ then $a^n \equiv b^n \pmod{m}$ for all positive integers n. Show by an example that the converse is not true 3

14. Solve the linear congruences
(i) $7x \equiv 4 \pmod{5}$ (ii) $7x \equiv 4 \pmod{15}$ 3

15. Solve the System of linear congruences
(i) $x \equiv 2 \pmod{9}$, $x \equiv 5 \pmod{6}$
(ii) $x \equiv 2 \pmod{5}$, $x \equiv 3 \pmod{7}$, $x \equiv 5 \pmod{8}$ 4

16. Use Euler's theorem to find the digit in the unit place of 3^{120} 3

17. Find the number of zeros at the end of $125!$

B. Complex Numbers

1. If z_1, z_2, z_3 be three complex numbers representing three vertices of a triangle which is equilateral then $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ and conversely 5

2. z is a variable complex number such that $\left|\frac{z-z_1}{z-z_2}\right| = K$ where z_1, z_2 are fixed complex numbers and K is a constant. Show that z lies on a circle in the complex plane if $K \neq 1$ and z lies on a straight line if $K = 1$ 4

3. If z_1 and z_2 are two complex numbers, then prove that $|z_1 - z_2|^2 \le (1+K)|z_1|^2 + (1+K^{-1})|z_2|^2$ for all positive real K 3

4. If $|z| \geq 3$ determine the least value of $\left|z + \frac{1}{z}\right|$ and the corresponding z 4

5. If z_1, z_2, z_3 are complex numbers and p, q, r are real numbers such that
$$\frac{p}{|z_2 - z_3|} = \frac{q}{|z_3 - z_1|} = \frac{r}{|z_1 - z_2|}$$
Prove that

$$\frac{p^2}{z_2 - z_3} + \frac{q^2}{z_3 + z_1} + \frac{r^2}{z_1 - z_2} = 0$$ 4

6 If z_1, z_2 are two complex numbers prove that

(i) $|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$ 3

(ii) $|1 + \bar{z}_1 z_2|^2 + |z_1 + z_2|^2 = (1 + |z_1|^2)(1 + |z_2|^2)$

7. Find the complex number z with maximum and minimum possible values of $|z|$ satisfying 4

(i) $\left|z + \frac{1}{z}\right| = 1$ (ii) $\left|z + \frac{4}{z}\right| = 3$

8. If $a = \cos x + i \sin x$, $b = \cos y + i \sin y$, $c = \cos z + i \sin z$ (x, y, z are real) and $a + b + c = abc$, prove that $\cos(x - y) + \cos(y - z) + \cos(z - x) = -1$ 4

9. If n be a positive integer and $(1+z)^n = p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n$

prove that

(i) $p_0 + p_4 + p_8 + \cdots = 2^{n-2} + 2^{\frac{n-2}{2}} \cos \frac{n\pi}{4}$

(ii) $p_0 + p_3 + p_6 + \cdots = \frac{2}{3}\left(2^{n-1} + \cos \frac{n\pi}{3}\right)$

(iii) $p_1 + p_4 + p_7 + \cdots = \frac{2}{3}\left(2^{n-1} + \cos (n-2)\frac{\pi}{3}\right)$ 5

10. Solve the equation $(x+1)^8 = (x-1)^8$ 4

11 In a triangle ABC, prove that $a^3 \exp(3iB) + 3a^2 b \exp(i(2B - A)) + 3ab^2 \exp(i(B - 2A)) + b^3 \exp(3iA) = c^3$

12 Prove that $x^n + 1 = (x+1) \prod_{k=0}^{(n-3)/2} \left[x^2 - 2x \cos \frac{(2k+1)\pi}{n} + 1\right]$, if n be an odd positive integer

Deduce that (i) $\sin \frac{\pi}{18} \sin \frac{3\pi}{18} \sin \frac{5\pi}{18} \sin \frac{7\pi}{18} = \frac{1}{16}$

(ii) $\cos \frac{\pi}{18} \cos \frac{3\pi}{18} \cos \frac{5\pi}{18} \cos \frac{7\pi}{18} = \frac{3}{16}$ 5

13. If z, a, b are complex numbers and $(ab)^z = a^z b^z$ but $ab \neq 0$ then prove that p.v. of $(ab)^z \neq$ (p.v. of a^z) (p.v. of b^z) 4

14. Find the general and principal values of 4

(i) $(1 - i)^{1+i}$ (ii) $(1+i)^{1-i}$ (iii) $i^{\log(1+i)}$

15. Express $\log(1 + \cos 2\theta + i \sin 2\theta)$, $\frac{\pi}{2} < \theta < \pi$ in the form $a + ib$ where a, b are real 3

16. Find the general solutions of
(i) $\sin z = 2i$ (ii) $\cos z = 2i$, (iii) $\cos z = -2$ 4
(iv) $\sinh z = 2$ (v) $\cosh z = 2$

17. If $\cos^{-1}(u+iv) = p+iq$, where u, v, p, q are real, prove that $\cos^2 p$ and $\cosh^2 q$ are the roots of the equation $x^2 - x(1+u^2+v^2) + u^2 = 0$ 4

18. If x be a real number, prove that $\cos^{-1}(ix) = 2n\pi \pm \left[\frac{\pi}{2} - i\log(x+\sqrt{x^2+1})\right]$, 4
n being an integer

19. Prove that the general solution of $\sin z = \cosh 4$ is $(4n+1)\frac{\pi}{2} \pm 4i$, n being an integer 4

20. Show that $\tan^{-1}(1+i) = \frac{1}{2}\left[(2n+1)\pi + \tan^{-1}(-2)\right] + \frac{i}{4}\log 5$, n being an integer 4

C Inequalities

1. If a, b, c, x, y, z be all real numbers and $a^2+b^2+c^2 = 1$, $x^2+y^2+z^2 = 1$ prove that 4
$-1 \leq ax+by+cz \leq 1$

2. If n be a positive integer greater than 2, prove that $(n!)^2 > n^n$ 3

3. Prove that $1!\,3!\,5! \cdots (2p-1)! > (p!)^p$ 3

4. If a, b, c, d be all real numbers, prove that $(a^2+b^2+c^2+d^2)(a^2+b^2)(c^2+d^2) \geq (abc+bcd+cda+dab)^2$ 4

5. If n be a positive integer, prove that 4
$\frac{1}{2\sqrt{n}} < \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$

6. If $a_1, a_2, \ldots, a_n$ be all positive real numbers and $a_1 a_2 \cdots a_n = K^n$ $(K>0)$, prove that 5
$(1+a_1)(1+a_2)\cdots(1+a_n) \geq (1+K)^n$

7. If A and G be the A.M. and G.M. respectively of n positive real numbers $a_1, a_2, \ldots, a_n$ prove that for $K>0$
$(K+A)^n \geq (K+a_1)(K+a_2)\cdots(K+a_n) \geq (K+G)^n$ 5

8. If a, b, c are positive rational numbers, prove that
$a^a b^b c^c \geq \left(\frac{a+b}{2}\right)^{(a+b)/2}\left(\frac{b+c}{2}\right)^{(b+c)/2}\left(\frac{c+a}{2}\right)^{(c+a)/2} \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}$ 5

9. If a, b, c, d are all positive real numbers and $s = a+b+c+d$, prove that

$$\frac{16}{s} \leq \frac{3}{s-a} + \frac{3}{s-b} + \frac{3}{s-c} + \frac{3}{s-d} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$$ 4

10. If $a_1, a_2, a_3, \ldots, a_n$ be n positive rational numbers and $s = \sum_{i=1}^{n} a_i$, prove that

$$\left(\frac{s}{a_1} - 1\right)^{a_1} \left(\frac{s}{a_2} - 1\right)^{a_2} \cdots \left(\frac{s}{a_n} - 1\right)^{a_n} \leq (n-1)^s$$ 4

11. If $a_1, a_2, \ldots, a_n$ are n positive real numbers in harmonic progression prove that 4

$$a_1 a_2 \cdots a_n > \left(\frac{2a_1 a_n}{a_1 + a_n}\right)^n$$

12. Find the greatest value of xyz where x, y, z are positive real numbers and

(i) $x^2 + y^2 + z^2 = 14$, (ii) $xy + yz + zx = 81$

13. Prove that the least value of $x + y + z$ is $4\sqrt{3}$ where x, y, z are positive real numbers satisfying the condition $x^2 y^3 z = 8$ 4

14. If n be a positive integer > 1, prove that

$$\left(\frac{n+1}{2}\right)^{\frac{n(n+1)}{2}} \leq 1^1 2^2 3^3 \cdots n^n < \left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}}$$ 5

15. If $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, prove that 8

$$\left(\frac{n - s_n}{n-1}\right)^{n-1} > \frac{1}{n}, \text{ if } n \geq 2.$$

D. Theory of Equation

1. Find the remainder when $x^4-3x^3+2x^2+x-1$ is divided by x^2-4x+3 3

2. Prove that x^2+px+p^2 is a factor of $(x+p)^n-x^n-p^n$ if n be odd and not divisible by 3.

3. If $f(x) = x^5-5x^4+12x^3-6x^2+7x-5$ 3
then express $f(x+1)$ as a polynomial in x

4. State the "Fundamental Theorem of Classical Algebra" and use it to prove that an algebraic equation of degree n has n roots and no more. 5

5. Form a biquadratic equation with rational coefficients, two of whose roots are $\sqrt{2} \pm 3$ 4

6 Find the values of a for which the equation $ax^3-6x^2+9x-4=0$ may have multiple roots and solve the equation in each case. 5

7. The equation $x^4-2x^3+18x^2-18x+81=0$ has four distinct roots of equal moduli. 5
Solve it.

8. Prove that the roots of the equation $\frac{A_1}{x+a_1}+\frac{A_2}{x+a_2}+\cdots+\frac{A_n}{x+a_n}=x+b$, are all real, where A_i, a_i, b are all real and $A_i>0$ for all $i=1,2,\ldots,n$. 4

9. Show that the equation $x^4-14x^2+24x+K=0$ has four real and unequal roots if $-11<K<-8$ Discuss the cases when $K=-8$, $K=11$ 5

10. State Descartes' rule of signs and apply it to find the nature of the roots of the equation $x^4+4x^3-x^2-10x+3=0$ 4

11. Use Sturm's functions to show that the roots of the equation $x^4+5x^3-13x+5=0$ are all real and distinct. 5

12. Solve the equation $4x^4+20x^3+35x^2+25x+6=0$ given that the roots are in arithmetic progression. 5

13. Find the relation among the coefficients of the equation $x^4+px^3+qx^2+rx+s=0$ if its roots α,β,γ and δ are connected by the relation 4
(i) $\alpha\beta+\gamma\delta=0$ (ii) $\alpha\beta+1=0$

14 Solve the equation $3x^4+20x^3-70x^2-60x+27=0$ given that the roots are in geometric progression. 4

15. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of
(i) $\Sigma \alpha^3$ (ii) $\Sigma \alpha^4$ (iii) $\Sigma \alpha^2 \beta \gamma$ (iv) $\Sigma \alpha^3 \beta^3$
(v) $(\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)(\alpha+\beta-2\gamma)$ 4

16. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, find the value of
(i) $\Sigma \frac{\alpha}{\beta}$ (ii) $\Sigma \frac{\alpha\beta}{\gamma}$ (iii) $\Sigma \frac{\alpha\beta}{\gamma^2}$ (iv) $\Sigma \frac{\alpha^2}{\beta}$ 5

17. If $\alpha + \beta + \gamma + \delta = 0$, prove that 5
$$\frac{\Sigma \alpha^5}{5} = \frac{\Sigma \alpha^3}{3} \cdot \frac{\Sigma \alpha^2}{2}$$

18. If $\alpha+\beta+\gamma = 1$, $\alpha^2+\beta^2+\gamma^2 = 3$, $\alpha^3+\beta^3+\gamma^3 = 7$, find the value of $\alpha^4+\beta^4+\gamma^4$. 4

19. If α, β, γ be the roots of the equation $x^3 + px^2 + qx + r = 0$, $(r \neq 0)$ find the equation whose roots are $\frac{1}{\beta^2} + \frac{1}{\gamma^2} - \frac{1}{\alpha^2}$, $\frac{1}{\gamma^2} + \frac{1}{\alpha^2} - \frac{1}{\beta^2}$, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} - \frac{1}{\gamma^2}$ 5

20. If α, β, γ are the roots of the equation $x^3 + qx + r = 0$ $(r \neq 0)$, find the equation whose roots are $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$, $\frac{\beta}{\gamma} + \frac{\gamma}{\beta}$, $\frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}$. 5

21. Transform the equation $x^4 + 4x^3 + 9x^2 + 10x - 6 = 0$ into one which shall want the second term and hence solve the given equation. 5

22. Obtain the equation whose roots are squares of the roots of the equation $x^4 - x^3 + 2x^2 - x + 1 = 0$ and use Descartes' rule of signs to the resulting equation to deduce that the given equation has no real root 5

23. If α be a root of the equation $x^3 + 3x^2 - 6x + 1 = 0$ prove that the other roots are $\frac{1}{1-\alpha}$ and $\frac{\alpha-1}{\alpha}$ 4

24. Define Reciprocal equation. Reduce the reciprocal equation $x^7 + 4x^6 + 4x^5 + x^4 - x^3 - 4x^2 - 4x - 1 = 0$ to a reciprocal equation of the standard form and solve it. 5

25. If $\omega_1, \omega_2, \ldots, \omega_m$ be m distinct mth roots of unity, prove that
$$(a + b\omega_1)^m + (a + b\omega_2)^m + \cdots + (a + b\omega_m)^m = m(a^m + b^m)$$ 4

26. Define special root of $x^n - 1 = 0$. Find the special roots of $x^{15} - 1 = 0$. Deduce that $2\cos\frac{2\pi}{15}$, $2\cos\frac{4\pi}{15}$, $2\cos\frac{8\pi}{15}$, $2\cos\frac{16\pi}{15}$ are the roots of the equation $x^4 - x^3 - 4x^2 + 4x + 1 = 0$ 5

27. If α be a special root of $x^8 - 1 = 0$, prove that $1 + 3\alpha + 5\alpha^2 + \dots + 15\alpha^7 = \frac{16}{\alpha - 1}$ 4

28. Find the equation whose roots are squares of the differences of the roots of the equation $x^3 + 3x^2 - 24x + 28$ 5

29. Solve by Cardan's method

(i) $x^3 + 3x^2 - 3 = 0$ (ii) $x^3 - 12x + 8 = 0$

(iii) $2x^3 - 3x + 1 = 0$ (iv) $x^3 - 6x^2 - 6x - 7 = 0$ 5

30. Solve by Ferrari's method

(i) $x^4 - 2x^2 + 8x - 3 = 0$ (ii) $x^4 + 4x^3 - 6x^2 + 20x + 8 = 0$

(iii) $2x^4 + 6x^3 - 3x^2 + 2 = 0$ (iv) $x^4 + 11x^2 + 10x + 50 = 0$

X

Matrix - I

1. Find the 3×3 matrices A and B when $A+B = 2B^t$ and $2A+3B = 6I_3$; $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ 3

2. Show that $(I_3 - A)(I_3 + A)$ is a symmetric matrix when A is a 3×3 symmetric or a skew symmetric matrix. 3

3. If A is a skew-symmetric matrix, show that the matrix A^2 is symmetric. 2

4.

4. Prove that if A and B are two matrices such that $AB = A$ and $BA = B$, then A and B are idempotent. 3

5. If A and B be two matrices such that $AB = 0$, can we deduce that either A or B is a zero matrix? Justify your answer. 2

6. Consider the matrix $A = \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix}$

   Prove that for any positive integer n, $A^n = A$ Give example of two non-null real matrices of order 3 whose matrix product is a null matrix. 4

7. For two matrices A and B for which the product AB is defined, show that $(AB)^T = B^T A^T$ (A^T denotes the transpose of A) 3

8. Express A as $P+Q$ when P is a symmetric matrix and Q is a skew-symmetric matrix, when $A = \begin{pmatrix} 2 & 1 & 0 & 3 \\ 5 & 3 & 2 & 8 \\ 4 & 2 & 1 & 5 \\ 1 & 6 & 7 & 0 \end{pmatrix}$ 3

9. define an idempotent matrix
   If A be an idempotent matrix of order 'n' show that $(I_n - A)$ is also idempotent.

10. If A be a symmetric matrix of order m and P be an $m\times n$ matrix, prove that $P^t A P$ is a symmetric matrix.

11. If A be a real skew-symmetric matrix of order n and P be a real $n\times 1$ matrix, prove that $P^t A P = 0$

12. Find all 2×2 matrices that commute with the real matrix
   (i) $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$; (ii) $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$, $bc \neq 0$; (iii) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $bc \neq 0$

13. Find all non-null real matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $A^2 = 0$

14. Find all real matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $A^2 = I_2$

2 15. If $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$, show that $A^3 - 6A - 9I_3 = 0$ Hence obtain a matrix B such that $BA = I_3$

3 16. If $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ show that $A^2 - 2A + I_2 = 0$. Hence find A^{50}

4 17. Prove that the equality $AB - BA = I_2$ cannot hold, whatever the real 2×2 matrices A, B may be.

MATRIX – II

1. If $A = \begin{bmatrix} a^2 & 2 & 2 & 2 \\ 4 & b^2 & 3 & 3 \\ 6 & 6 & c^2 & 5 \\ 7 & 7 & 7 & 7 \end{bmatrix}$, Show that A is a non singular matrix, for $a, b, c \in Q$ (Set of rational numbers). 5

2. A is a 3×3 non-null matrix (real) and $A^2 - A - I_3$ is a null matrix. Show that A^{-1} exists and $A^{-1} = A - I_3$.

3. Show that the equations
$$x+2y-z=5$$
$$3x-y+2z=3$$
$$ax+y+z=2a$$
have unique solutions for $a \neq 4$. Solve the equations (by Determinant or Matrix Inversion method) when $a=1$. 2+3

4. Find matrices P and Q such that PAQ is a matrix in the normal form when $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$

5. If $A = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 3 & 0 \\ 1 & 4 & 0 \end{bmatrix}$, find $\det(\mathrm{adj}.A)$ 1

6. Let A be an $n\times n$ matrix whose elements are complex numbers. Show that $A+\mathring{A}$ is Hermitian, $\mathring{A} = (\bar{A})^t$ 2

7. If a, b, c are real and unequal, find the rank of the matrix
$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$ when (i) $a+b+c=0$; (ii) $a+b+c \neq 0$

8. Find different rational values of a and b for which the equations
$$x+y+z=1$$
$$2x+3y+5z=b$$
$$4x+5y+az=b^2$$ have (i) unique solution, (ii) more than one solution, (iii) no solution. 5

9. If A be an invertible matrix, prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$, where A^t denotes the transpose of A.

10. Solve, by matrix inversion method, the following system of equations:

$$x + 2y + 3z = 14$$
$$2x - y + 5z = 15$$
$$-3x + 2y + 4z = 13$$

11. Reduce the real quadratic form $2x^2 + 2y^2 + 5z^2 - 4xy - 2xz + 2yz$ to its normal form and find its rank and signature. 5

12. If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix}$, find a matrix B such that $AB = 6I_3$

Hence solve the following system of equations

$$2x + y + z = 5$$
$$x - y = 0$$
$$2x + y - z = 1$$

3+2

13. Obtain the fully reduced normal form of the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix}$$

and hence find the rank of the matrix

4+1

14. If A and B are two $n \times n$ matrices and A has an inverse, then show that $(A+B)A^{-1}(A-B) = (A-B)A^{-1}(A+B)$ 2

15. Let A be a square matrix of order n. Prove that $A.\mathrm{Adj}A = \mathrm{Adj}A.A = \det A \cdot I_n$, where I_n is the identity matrix of order n 5

16. Let A be a skew-symmetric matrix and $\det A \neq 0$. Prove that the order of A must be even. 3

17. Reduce the real quadratic form $xy + yz + zx$ to its normal form. Find its rank and signature. 5

18. Determine the values of a and b so that the system of equations

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

admits of (i) only one solution, (ii) no solution, (iii) infinite number of solutions.

19. Let A be a matrix such that $(I+A)$ is non-singular. Show that A is skew-symmetric if $B = (I-A)(I+A)^{-1}$ is orthogonal. 5

20 For the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, find a matrix P such that $P^{-1}AP$ is a diagonal matrix. 5

21. A and B are real orthogonal matrices of the same order and $\det A + \det B = 0$. Show that $A+B$ is a singular matrix. 5

22. When is a real quadratic form said to be positive definite? Show that the quadratic form $5x^2+y^2+10z^2-4yz-10zx$ is positive definite. 5

23. Find an orthogonal matrix of order 3 on the set of integers whose first row is $(1,2,-1)$. 5

24. Determine the values of a and ~~B~~ b so that the system of equations

$$\begin{aligned} x+2y+z &= 1 \\ 3x+y+2z &= b \\ ax-y+4z &= b^2 \end{aligned}$$

has (i) unique solution, (ii) no solution; (iii) many solutions in the field of real numbers 5

25. Find the inverse of the matrix (if it exists.)

$$A = \begin{pmatrix} 1 & 3 & 2 & 5 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{pmatrix}$$ 5

26. When is a real quadratic form said to be positive definite? Prove that the real quadratic form $ax^2+bxy+cy^2$ is positive definite if and only if $a>0$ and $b^2<4ac$ $(a,b,c \neq 0)$. 5

27. If the sum of the elements in each row of a non-singular matrix is $K\,(\neq 0)$, then show that the sum of the elements in each row of the inverse matrix is K^{-1} 5

28. If A and C are two non-singular matrices of the same order and if $D = CAC^{-1}$ and λ is any scalar, prove that

$$\det(D+\lambda I) = \det(A+\lambda I)$$ 5

29 Find the matrix A of the quadratic form

$$x^2+2y^2-7z^2-4xy+8zx$$ 5

30. Prove that every non-singular matrix can be expressed as a product of finite number of elementary matrices. 5

31. determine the rank of the matrix 5

$$\begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & 2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ 1 & 7 & -4 & 1 \end{bmatrix}$$

32. Find a non-singular matrix P such that $P^t A P$ is a diagonal matrix, where A is (i) $\begin{pmatrix} 2 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 10 \end{pmatrix}$; (ii) $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix}$; 5+5

33. Find a non-singular matrix P s.t. $P^t A P$ is the normal form of A under ~~conve~~ congruence 5+5

(i) $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$; (ii) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

34. Examine if the matrices A and B are congruent when

$A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 & 1 \\ 4 & 9 & 4 \\ -1 & 4 & 2 \end{pmatrix}$ 5

35. Find the rank and signature of the following symmetric matrix.

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$ 5

36. use elementary row operation on A to obtain A^{-1} when 5

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

37. Express A as the product of elementary matrices, when

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{pmatrix}$$ 5

38. Obtain non-singular matrices P and Q such that ~~PAQ~~ $PAQ = R$, where R is the fully reduced normal form of the matrix $A = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 0 \\ 3 & 1 & 6 & 3 \end{pmatrix}$ 5

39 If $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, find the rank of the matrix $A + A^2 + A^3$

40. Prove that the rank of a real skew-symmetric matrix cannot be 1. 3

41. If the rank of a real symmetric matrix be 1 show that the diagonal elements of the matrix cannot be all zero

42. A is a real and non-symmetric of order 3. Prove that the rank of the matrix $A - A^t$ is 2.

43. A is a non-singular matrix of order 4 determine the rank of the matrix (i) A^{-1}, (ii) A^3, (iii) adj A, (iv) $2A$; 5×4

44. If $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$, show that A^{-1} exists

Without computing A^{-1} find the sum of the elements of A^{-1} 5

45 If $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$, show that $A^2 - 10A + 16I_3 = 0$

Hence obtain A^{-1}

46. P is an $n \times n$ real orthogonal matrix with $\det P = -1$. Prove that $P + I_n$ is a singular matrix

47. If A is a real ~~or~~ orthogonal matrix and $I + A$ is non-singular, prove that the matrix $(I+A)^{-1}(I-A)$ is skew-symmetric

48. If $(I+A)^{-1}(I-A)$ is a real orthogonal matrix, prove that the matrix A is skew-symmetric. 5

49. If A is a real skew-symmetric matrix and $I+A$ is non-singular, prove that the matrix $(I+A)^{-1}(I-A)$ is orthogonal.

50. Let A be a square matrix s.t. $I+A$ is non-singular. Let $\bar{A} = (I+A)^{-1}(I-A)$. Prove that (i) $I+\bar{A}$ is non-singular
(ii) $\bar{\bar{A}} = A$ 5+5

51. A and B are non-singular matrices s.t. $AA^t = BB^t$. Prove that there exist orthogonal matrices P, Q s.t. $A = BP$ and $B = AQ$. 5

52. A non-singular matrix P commutes with P^t. Prove that P^t commutes with P^{-1} and the matrix $P^{-1}P^t$ is orthogonal. 5

53. Find a real orthogonal matrix of order 3 having the elements $\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ as the elements of a row. 5

54. Find a real orthogonal matrix of order 3 having the elements $\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}$ as the elements of a column. 5

55. Express the matrix $A = \begin{pmatrix} 1 & 2+i & 1-i \\ 2-i & 1+2i & 3 \\ 2+i & 2 & 1+i \end{pmatrix}$

as the sum of a Hermitian matrix and a skew-Hermitian matrix.

56. Prove that the matrix $\frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$ is orthogonal.

Utilise this to solve the eqns. 5

$$\begin{aligned} x-2y+2z &= 2 \\ 2x-y-2z &= 1 \\ 2x+2y+z &= 7 \end{aligned}$$

57. If $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$, show that $AB = 6I_3$

Utilise this result to solve 5+5

(i) $\begin{aligned} 2x+y+z &= 5 \\ x-y &= 0 \\ 2x+y-z &= 1 \end{aligned}$; (ii) $\begin{aligned} x+y+3z &= 6 \\ 2x-4y &= 0 \\ x+y-3z &= 0 \end{aligned}$

DETERMINANTS

1. Define a skew-symmetric determinant. Show that such a determinant of order four is a perfect square. 1+4

2. Prove that $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$,

   when $\omega^3 = 1$.

3. Let a, b, c be three unequal real numbers. Show that

$$\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix} \text{ is never zero}$$

~~4~~ Is the result valid, when a, b, c are complex numbers? 4+1

4. Let a, b, c be non-zero real numbers. Show that

$$\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(ab+bc+ca)^3$$ 3

5. Let $D_n = \begin{vmatrix} 1 & x & 0 & 0 & & 0 \\ x & 1 & x & 0 & 0 & 0 \\ 0 & x & 1 & x & \cdots & 0 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & x & 1 \end{vmatrix}$ be a determinant of order n in whic the diagonal elements are 1 and those just above and just below the diago elements are x and all other elements are zero.

   Prove that $D_4 - D_3 + x^2 D_2 = 0$. Hence prove that

$$\begin{vmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{vmatrix} = 5/16.$$

6. Expanding by Laplace's method, show that

$$\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2$$

7. Let $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$ and $\Delta' = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ be its adjoint.

Prove that $\dfrac{BC - F^2}{a} = \dfrac{CA - G^2}{b} = \dfrac{AB - H^2}{c} = \Delta$ 5

8. If $\Delta = \begin{vmatrix} h & a & 0 \\ \frac{1}{h} & \frac{1}{b} & \frac{1}{f} \\ 0 & c & f \end{vmatrix}$ and $\Delta' = \begin{vmatrix} \frac{1}{bc} - \frac{1}{f^2} & -\frac{1}{ch} & \frac{1}{fh} \\ af & -fh & ch \\ \frac{1}{fh} & -\frac{1}{af} & \frac{1}{ab} - \frac{1}{h^2} \end{vmatrix}$,

find the value of $\dfrac{\Delta'}{\Delta^2}$ in its simplest form. 5

9 Prove that $\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3(s-a)(s-b)(s-c)$

when $2s = a+b+c$

10. If $D = \begin{vmatrix} ap & aq & bp & bq \\ ar & as & br & bs \\ cp & cq & dp & dq \\ cr & cs & dr & ds \end{vmatrix} = \begin{vmatrix} a & b & 0 & 0 \\ 0 & 0 & a & b \\ c & d & 0 & 0 \\ 0 & 0 & c & d \end{vmatrix} \times A$

then find the determinant A. Hence evaluate the determinant D. 5

11. Let a_{ij} be an $n \times n$ non-singular matrix. If A_{ij} be the co-factor of a_{ij} in $\det(a_{ij})$, $i,j = 1,2,\ldots n$, prove that $\det(A_{ij}) = [\det(a_{ij})]^{n-1}$, 5

12. Prove that $\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix} = a_1 a_2 a_3 a_4 \left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}\right)$.

when $a_i \neq 0$, $i = 1,2,3,4$

13. Solve the following system of equations by Cramer's rule. 5

$$\begin{aligned} x+2y-3z &= 1 \\ 2x-y+z &= 4 \\ x+3y &= 5 \end{aligned}$$

14. Prove that $\begin{vmatrix} 1+x^2 & x & 0 & 0 \\ x & 1+x^2 & x & 0 \\ 0 & x & 1+x^2 & x \\ 0 & 0 & x & 1+x^2 \end{vmatrix} = 1+x^2+x^4+x^6+x^8$ 5

15. Determine the value of $\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix}$ by Laplace's expansion in terms of minors of order 2 obtained from the first two rows.

16. Let $A = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and $A, B, C, \ldots$ be the co-factors of $a, b, c, \ldots$ respectively in Δ

Prove that $\dfrac{BC-F^2}{a} = \dfrac{GH-AF}{f} = \Delta$ 5

17. Express the determinant $\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$

as a square of a determinant of order 3. Hence determine the value of Δ.

5

18. Evaluate the following determinant by expressing it as a product of two determinants.

$$\begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix}$$ 5

19. Prove that $\begin{vmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ a^2 & a^3 & 1 & a \\ a & a^2 & a^3 & 1 \end{vmatrix} = (1-a^4)^3$ 5

20. Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$ 5

21. For an $n \times n$ matrix A, prove that adj det $A = [\det A]^{n-1}$ 5

22. Show that $\begin{vmatrix} a & b & c & 0 \\ b & a & 0 & c \\ c & 0 & a & b \\ 0 & c & b & a \end{vmatrix} = (a+b+c)(a+b-c)(a-b+c)(a-b-c)$ 5

23. Prove that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \binom{m}{1} & \binom{m+1}{1} & \binom{m+2}{1} & \binom{m+3}{1} \\ \binom{m+1}{2} & \binom{m+2}{2} & \binom{m+3}{2} & \binom{m+4}{2} \\ \binom{m+2}{3} & \binom{m+3}{3} & \binom{m+4}{3} & \binom{m+5}{3} \end{vmatrix} = 1$ 5

when m is a positive integer and $\binom{n}{r} = {}^nc_r$.

24. Prove that $\begin{vmatrix} a^2+x & ab & ac & ad \\ ab & b^2+x & bc & bd \\ ac & bc & c^2+x & cd \\ ad & bd & cd & d^2+x \end{vmatrix} = x^3(a^2+b^2+c^2+d^2+x)$ 5

25. Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$ 5

26. Prove that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^4 & \beta^4 & \gamma^4 & \delta^4 \end{vmatrix} = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)(\alpha+\beta+\gamma+\delta)$ 5

27. Prove that $\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \\ \alpha^4 & \beta^4 & \gamma^4 & \delta^4 \end{vmatrix} = (\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\alpha\beta\gamma)$ 5

28. Prove that $\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6$ 5